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The present investigation attempts to solve some problems which were interesting to me, and which did not seem to be satisfactorily treated in such works as I could conveniently consult. It is possible that additional research among mathematical treatises would have proved more successful; but in any case, it is not probable that the precise method which I have here employed would have been adopted by previous writers. The plan of inquiry, therefore, may be new, even if its conclusions only confirm older discoveries.

Under the laws of inertia and gravitation, a particle of negligible mass circulates in one of the conic sections about a central point from which emanates an attraction proportional to the inverse square of its distance. Denoting this variable distance by R and its inverse square by R^{-2}, the question may be asked what effects will follow if we substitute for -2 an exponent n to which we may assign any value. Various cases of this kind are considered in ordinary text-books, especially that in which $n = +1$, but some which seem particularly interesting are evaded. In attempting the solution of these, I have been led to the general principles to be stated below.

I.

Transverse motion and the force derived from it.

When the moving particle proceeds directly toward or directly from the central fixed point, its course is controlled only by the velocity attributed to it and by a single force. In other cases, its motion may be resolved into radial and transverse components perpendicular to each other. Its transverse motion produces a second force, called centrifugal by established usage only when the orbit described is a circle. If not, a distinctive name for this second force is required, and if such a name is to be derived from an ancient language, the words apocentric or divellent might be suggested. But, in the present

discussion, it will be sufficient to employ the words outward and inward for forces tending to increase or to diminish the distance between the moving particle and the central point.

The outward force is left without a name in the usual treatises upon our present subject. They make it clear, however, that such a force exists, and that it varies inversely to the cube of the radius vector. For instance, we find in Price's Treatise on Infinitesimal Calculus, Vol. III, p. 462, the equation $d^2r/dt^2 - h^2/r^3 = - P$, in which P denotes a central force, and h denotes the constant $r^2d\theta/dt$ resulting from the law of inertia. This equation, in the form $d^2r/dt^2 = h^2/r^3 - P$, denotes the excess of a force tending to elongate the radius vector r, and inversely proportional to r^3, over any force tending to decrease r. If $h^2/r^3 < P$, the inward force is the stronger.

Since it will also be convenient to have an expression for the transverse motion in terms of the outward force, a geometrical proof of the principle just stated will here be added.

In Fig. 1, let C represent the fixed point about which the moving particle describes its orbit, and AB the straight line through which that particle moves under the law of inertia during one of the intervals between successive applications of force ultimately to be supposed continuous and consequently infinitesimal in the second degree. From B let fall the perpendicular BD on CA; and at B erect a perpendicular on CB, intersecting CA, produced if necessary, at F. Since CB>CD, while CF>CB, there is a point E between D and F such that CE = CB. When AB becomes infinitesimal with respect to CA or CB, DE = EF, as is easily shown, so that DF = 2DE. The effect of outward force in elongating CD is represented by DE.

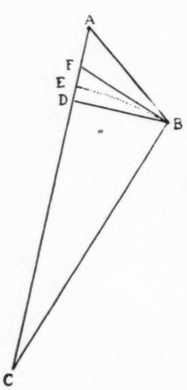

Fig. 1.

The transverse motion represented by BD is inversely proportional to CD, according to the fundamental law of equality of areas, which need not here be repeated. The squared transverse motion is therefore inversely proportional to the squared radius vector, which is equal to the product of CD and DF. Accordingly, DF, and consequently DE, is proportional to the inverse cube of CD or CA, which are ultimately equal; and the transverse motion is expressed by the square root of the product of the radius vector and twice the

outward force. Any reader familiar with the elements of orbital motion will readily fill up the outline of this demonstration by adding details which it seems needless to state here.

II.

The relation of outward and inward force, when $n > -3$.

The present discussion will be confined to those cases in which the exponent n, defining the law of inward force, as above stated, is at least not less than -3. Under this restriction, there may be, and if $n > -3$, there must be, with the exceptions noticed at the end of section IX, some radius vector, to be denoted by P, at which the outward and inward forces are equal. Under the law of gravitation, this radius vector is the semiparameter of the orbit, as appears from the ordinary treatment of the subject. For values of n other than -2, it is to be found by methods hereafter to be described. For the present, it is required only to show that it will occur.

Suppose that in a particular case, and at a given moment, the inward force is the stronger, while at the same time the radius vector is diminishing by the operation of the law of inertia. The momentary effects of the forces are both infinitesimals of the second order, so that a definite ratio may exist between them. Let this ratio be denoted by a, so that the momentary effect of the inward force may be a times as great as that of the outward force. Let the inward force vary directly as that power of the distance R denoted by the exponent n, and let $n > -3$. At a later time, the ratio of the forces will be expressed by $aR^n/R^{-3} = aR^{n+3}$, in which $n + 3 > 0$. Let R decrease without limit, and before it becomes equal to 0, R^{n+3} will become equal to $1/a$, since a is finite. The forces will then be equal.

At this time, the inward velocity will reach its maximum, since it has previously been increasing from an excess of inward force, and must afterwards decrease. If this velocity is finite, it must be extinguished, before $R = 0$, by the unlimited decrease of R^{n+3}.

If $n = -3$, the inward and outward forces are always equal or never equal. If $a > 1$, the inward force is always the stronger; if $a < 1$, it is always the weaker. The various results to be expected from such conditions have been discussed in previous treatises, for example, by Price (Treatise on Infinitesimal Calculus, Vol. III., p. 487), and will not be examined in the present article.

III.

Definitions of terms and symbols here to be employed.

The words periastron and apastron will be used in their ordinary senses of the situations in which the radius vector has its least or its greatest value. We have just seen that periastron will always occur in the cases here to be examined, but that apastron may never be attained. The straight lines drawn through the fixed point in the directions of periastron and apastron will be called the axes of periastron and apastron.

The geometrical quantities having special designations in the present discussion are R, radius vector; Q, least value of R; L, greatest value of R if such a value occurs; P, value of R at which the inward and outward forces are equal, and the inward or outward velocity is at a maximum; Y, variable distance measured from the fixed point, toward periastron, to the foot of the perpendicular U, let fall upon the axis of periastron from any position of the moving particle; Z, variable distance measured in the same manner and direction as Y, having the value 0 not at the fixed point, but at the foot of that value of U which occurs when $R = L$; K, greatest value of Z, occurring when $R = Q$; X, variable distance measured from the fixed point, toward apastron, to the foot of the perpendicular let fall upon the axis of apastron from any position of the moving particle; S, variable distance measured in the same manner and direction as X, having the value 0 when $R = Q$; H, greatest value of S, occurring when $R = L$; v, angle between the direction of R and the axis of periastron, having the value 0 when $R = Q$; v_0, the value of v when $R = L$. Hence $Y = Z + L \cos v_0$, and $X = S + Q \cos v_0$.

Designations of algebraic quantities are n, that exponent of R which shows the law of inward force under consideration, so that for the law of gravitation $n = -2$; m, infinitesimal of the second order, expressing the immediate effect of inward force in orbits of the same system; m', similar infinitesimal, showing the immediate effect of transverse motion in increasing R; e, a constant ratio between functions of R and Z, these functions to be determined for different values of n, and the value of e to be determined accordingly; w, an abbreviation for $\sqrt{(n+3)}$; e', the constant ew.

Since P denotes that value of R at which the inward and outward forces are equal, $mP^n = m'P^{-3}$, whence $m' = mP^{n+3}$. Hence for any

value of R, the outward force is expressed by mP^{n+3}/R^3, and the inward force by mR^{n+3}/R^3, so that $m(P^{n+3} - R^{n+3})/R^3$ expresses the excess of outward force if positive, and of inward force if negative. This expression, then, denotes the actual force momentarily existing at any value of R.

Fig. 2 illustrates the meaning of the geometrical terms above defined. It represents a curve of the form which will result from an inward force inversely proportional to the first power of the distance, so that $n = -1$. The fixed point, from which the inward force is assumed to emanate, is represented by C, the point of periastron by Q, and the

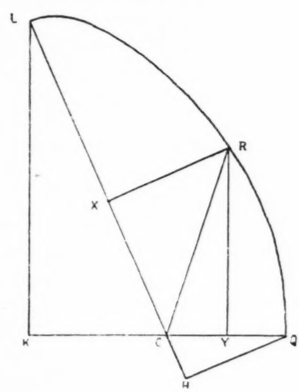

Fig. 2.

point of apastron by L. The point R is the assumed position of the moving particle. The line CR represents the radius vector R; CQ represents Q, and CL represents L. The position of the radius vector P has not been computed for this particular figure. The lines CY, RY, KY, KQ, CX, HX, HL, respectively represent the quantities Y, U, Z, K, X, S, H. The angle RCQ represents v, and LCQ v_0.

IV.

Law of variation of actual force.

The actual force, for any value of R assumed to be momentarily constant, will vary with respect to R^{n+3}, as has just appeared. When $n = -2$, $R^{n+3} = R$, and we know that in this case dR and dY

vary proportionally, while $w^2 = n + 3 = 1$. If $w > 1$, dR is affected by a force greater than that affecting dY. While R is decreasing from the value P, Y is increasing toward the value Q, so that the ratio dR/dY is negative. The variation tending to change it is $n + 3$ times as great as that which exists when $w = 1$, so that this variation is proportional to $(n + 3)\, dR/dY$. It may be desirable, however, to examine this result more minutely, since it is to be the basis of subsequent inquiry.

In discussing the effect of forces, we are obliged to consider them as applied, not continuously, but at intervals which we may ultimately regard as infinitesimal. During each of these intervals, the rates of variation, both of R and of Y, are assumed to be uniform; that is, t denoting time, dR/dt and dY/dt are constants. In the case of R, if no actual force exists, the outward and inward forces are equal. In the absence of force, the variation of Y is uniform by the law of inertia. In differentiating any function of R or of Y, we accordingly regard these quantities, for the moment, as independent variables.

Let any particular value of R be denoted by R_0, and regard it as constant for the moment. In the succeeding interval of time, we will suppose R_0 to be reduced to a smaller value R, so near R_0 that the difference is infinitesimal. To determine the variation of the force acting on dR when $R = R_0$, we differentiate the expression $m\,(R_0^{n+3} - R^{n+3})/R_0^3$ with respect to R, regarding R_0 as constant. The resulting differential coefficient is $-\,m(n + 3)R^{n+2}/R_0^3$, and disregarding the difference between R_0 and R, it becomes $-\,m(n + 3)\ R^{n-1}$. If we regard the denominator as R^3 instead of $R^3{}_0$, we have for the differential itself $-\,m[R^3(n + 3)R^{n+2} + 3R^2(R_0^{n+3} - R^{n+3})]\ dR/R^6$. The second term of the numerator is of a higher order of infinitesimals than the first, $(R_0^{n+3} - R^{n+3})$ being infinitesimal. The differential coefficient becomes as before $-\,m(n + 3)R^{n-1}$.

The variation of Y is not affected by any outward force, but is affected by the inward force on R in the ratio Y/R, as appears from consideration of the right triangle shown in Fig, 2, of which R is the hypotenuse while U and Y are the legs. The effect of inward force on R being $-\,mR^n$, its effect on Y is $-\,mR^nY/R_0 = -\,mR_0^{n-1}\,Y$ for any particular value of R denoted by R_0, which is to be regarded as constant for the moment. The differential coefficient is $-\,mR_0^{n-1}$, and we have $(n + 3)\ dR/dY$ for the ratio of the variations of force at any moment affecting dR and dY. By the definition of Z, $dY = dZ$, and the action of force upon Z is the same as its action on Y. The ratio of corresponding variations of force upon dR and dZ is therefore

$(n + 3)\, dR/dZ = w^2 dR/dZ$, the ratio dR/dZ being negative, since R decreases as Z increases.

We have now to determine the corresponding ratio of forces.

The action of a constant force upon a finite distance is proportional to the square of the time allowed for such action; and a force depending on the exponent of that distance is to be regarded as constant while we regard the distance as momentarily constant. This requires the time of action to be infinitesimal, and the squared time to be infinitesimal in the second order.

Similarly, the action of a constant variation of force upon the first differential of a distance is proportional to the squared time allowed for its action; and such a variation of force is to be regarded as constant while we regard that first differential as momentarily constant. This requires the time of action to be infinitesimal with respect to that first differential, and the squared time to be similarly infinitesimal in the second order. If we consider the time of action to be in the first instance infinitesimal in the second order, its square is infinitesimal in the fourth order. But the relative order of the two infinitesimals of time is in either case that of a quantity and its square.

In the absence of force, dR and dZ vary uniformly with time; and while the time allowed for their variation is the same for each, their ratio remains unaffected by the order of infinitesimals to which that time belongs.

In the expression for the ratio of variations of force, $w^2 dR/dZ$, the factor w^2 is equivalent to a ratio between the squared times allowed for the variations of dR and dZ, to whatever order of infinitesimals these times belong. For if the original ratio dR/dZ is multiplied by such a ratio between those squared times, the product will be w^2 times as great as before.

The square root of that ratio of squared times will be the ratio of the times by which dR and dZ are multiplied in the corresponding expression for the ratio of the forces acting upon them. This process is an integration merely in the sense that the integral of an infinite series of squared differentials of time is the first power of such a differential. The ratio dR/dZ is unaffected, as shown above, by the change thus indicated in the order of infinitesimals of time.

If, therefore, $w^2 dR/dZ$ represents a ratio of variations of force, the corresponding ratio of forces is $w dR/dZ$.

V.

Equations of curves derived from the preceding laws.

Let us first consider cases in which $w > 1$. In these, an apastron will ultimately occur. For, when $R > P$ and is increasing, the inward force is the stronger, and tends to retard the increase of R and the decrease of Z. But if $w > 1$, and the forces have the ratio wdR/dZ, the tendency to check the increase of R exceeds that checking the decrease of Z. Hence dR will ultimately become less than $-dZ$, however much it may have exceeded $-dZ$ when R had the value P; so that R will subsequently increase more slowly than Z decreases, Z will reach the value 0 before R reaches the value ∞, and, by the definition of Z, R cannot further increase, and must attain its greatest value L. The quantities K and L must then be regarded as finite, although we cannot set a definite limit to their increase.

When $w < 1$, the chance of an apastron diminishes. When $w = 0$, it has already appeared that R always or never has the value P.

We will now attempt to find functions of R and Z, the variations of which, when corrected for the corresponding forces, shall have a constant ratio.

The ratio wdR/dZ is a ratio of forces acting directly on dR and dZ, and indirectly on R and Z themselves. The corresponding ratio of forces acting directly on $R^{w-1}\,dR$ and $Z^{w-1}dZ$ is R^{w-1}/Z^{w-1} times as great, that is, $wR^{w-1}dR/Z^{w-1}dZ$. This is also the ratio of variations of wR^w and of Z^w, which is $w^2R^{w-1}dR/wZ^{w-1}dZ$, when we regard R and Z as varying uniformly, undisturbed by any force. The ratio of forces acting upon the variations is accordingly the same as the ratio of the variations themselves when undisturbed by force.

In all such cases, the existing ratio of the variations is not changed by the application of the forces, not merely at a single instant, when the forces are negligible as compared with the variations on which they act, but after they have developed accelerating or retarding effects upon the variations. Let A and B denote two velocities, a and b forces acting upon them. If $A/B = a/b = k$, so that $A = kB$ and $a = kb$, $(A + a)/(B + b) = k(B + b)/(B + b) = k$.

We conclude, therefore, that the variations of wR^w and of Z^w have a constant ratio e' when the forces acting upon them have the ratio wdR/dZ. Hence the variations of R^w and of Z^w have the constant ratio $e'/w = e$.

At periastron, the value of R is Q and that of Z is K. At other points, therefore, $(R^w - Q^w) = e(K^w - Z^w)$, which will be one equation of the curve described by the moving particle. When apastron occurs, $R = L$ and $Z = 0$, so that $e = (L^w - Q^w)/K^w$.

Unless e' is infinite, e tends to approach 0 as w increases, and the orbit ultimately becomes a circle. This is also evident from the consideration that, with w indefinitely large, any radial movement corresponding to a given transverse movement will instantly be checked by the unlimited increase of inward force if the radial movement is outward, or of outward force if that movement is inward.

A second form of the equation of the curve, when apastron occurs, is obviously $(L^w - R^w) = eZ^w$, in which, when $R = Q$, $Z = K$, and $e = (L^w - Q^w)/K^w$ as before.

Equations representing the curve may also be derived from the quantities X, S, and H, corresponding, when measures are made along the axis of apastron, to Y, Z, and K in the system above employed. The values of X and S, like those of Y and Z, vary uniformly in the intervals between successive applications of force, so that $dR/dX = dR/dS$, and is proportional to dR/dZ, but not equal to it except in a circular orbit.

By the method already explained, we find the ratio of the corresponding variations of force on R and on S to be $(n + 3)\, dR/dS$; the ratio of forces $w dR/dS$; and the resulting equations $(L^w - R^w) = e\,(H^w - S^w)$ and $R^w - Q^w = e\,S^w$. The value of e now becomes $(L^w - Q^w)/H^w$. Denoting the former value of e by e_1, and the new value by e_2, we find $e_2 = e_1 K^w/H^w$.

We found above $(L^w - R^w) = e_1 Z^w$, and we now find $(L^w - R^w) = e_2\,(H^w - S^w)$, whence $(K^w/H^w)\,(H^w - S^w) = Z^w$, and $H^w Z^w + K^w S^w = H^w K^w$. When $w = 2, H = A$, the semi-axis major; $K = B$, the semi-axis minor; $Z = Y$, and $S = X$. We then have the familiar equation $A^2 Y^2 + B^2 X^2 = A^2 B^2$. When $w = 1$, and apastron occurs, as is here assumed to be the fact, $H = L + Q$ and $K = Q + L$, the value of each being the major axis $2A$; $S = 0$ at periastron, and $Z = 0$ at apastron; the equation becomes $S + Z = 2A$.

By the substitution of $R \cos v$ for Y, we convert the equation $(L^w - R^w) = e\,Z^w$ into $L^w = R^w + e\,(R \cos v - L \cos v_0)^w$, but unless $w = 1$ or 2, this form is less convenient than those above stated.

VI.

Angles formed by the axes of periastron and apastron.

The substitution of $R \cos v$ for Y is advantageous, however, in finding the value of v_0, when $R \cos v$ is taken as the independent variable, since $dZ = dY$. Hence the corresponding changes in R and in $\cos v$ are reciprocal.

At periastron, $dR = dZ = 0$, and the ratio dR/dZ is indeterminate for the moment. The subsequent values of dR and dZ result from forces which have the ratio w, as was found above, in section IV. This ratio is independent of the order to which the infinitesimals of time belong, since the time allowed for the action of the forces is the same for dR as for dZ. The variations produced by these forces, and the consequent changes in R and Z themselves, retain this ratio.

While R varies from Q to L, Z varies from K to 0, and $(K - Z)$ from 0 to K. The change in R is w times the change in Z. Hence the corresponding change in $\cos v$ is $1/w$ as great as that in Y, which is equal to that in Z.

When $w = 1$, dR/dZ is constant, and the total variation of $\cos v$, from periastron to apastron, is 2, from $+1$ to -1. In other cases, the total variation of $\cos v$ is $2/w$. When $w = 2$, $2/w = 1$; hence $\cos v$ varies from $+1$ at periastron to 0 at apastron, and $v_0 = 90°$, as we know independently.

For values of w between 1 and 2, v_0 is obtuse, and $\cos v_0 = -(2/w - 1)$. For example, when the inward force is inversely proportional to the first power of the distance, $w = \sqrt{2}$, $\cos v_0 = -(\sqrt{2} - 1)$, and the approximate value of v_0 is $180° - 65°.5 = 114°.5$. When the inward force is constant, $w = \sqrt{3}$; $\cos v_0 = -(2/\sqrt{3} - 1)$, and the approximate value of v_0 is $180° - 81°.1 = 98°.9$.

When $w > 2$, v_0 becomes acute, and if w increases without limit, v_0 approaches 0. We have already seen that in this case the orbit is circular.

When $w < 1$, v_0 will exceed 180°, approaching 360° as w approaches 0. These cases will not here be minutely examined.

The value of K results from that of $\cos v_0$ by means of the equation $K = Q - L \cos v_0$, in which $\cos v_0$ is negative if v_0 is obtuse. The value of H is $L - Q \cos v_0$. The curve can accordingly be drawn when values of Q, L, and w are assumed.

When $w > 1$, $H > K$. For $H - K = (L - Q \cos v_0) - (Q - L \cos v_0) =$

$(L - Q)(1 + \cos v_0)$, and $\cos v_0$ cannot be less than -1. If $w = 1$, $H = K$, the value of either being the major axis of the ellipse if apastron occurs. If not, $L = \infty$, and $(L - Q)(1 + \cos v_0)$ becomes indeterminate. In either case, $e = (L - Q)/K = 2C/2A = C/A$, in which C and A have their usual meanings in the equations of a conic section, and e denotes the eccentricity.

When $w = 2$, $\cos v_0 = 0$, while H and K are respectively equal to the major and minor semi-axes of the ellipse. If the curve is referred to the axis of periastron, $e = (A^2 - B^2)/B^2$, in which A and B have their usual meanings of the two semi-axes, and e may have any value. If the curve is referred to the axis of apastron, $e = (A^2 - B^2)/A^2$, and cannot exceed 1, being the squared eccentricity of the ellipse.

VII.

Relative distances of points on the curve from the axis of periastron.

Since $R^w = (L^w - eZ^w)$ and $Y = Z + L \cos v_0$, while $R^2 = U^2 + Y^2$, $U^2 = (L^w - eZ^w)^{2/w} - (Z + L \cos v_0)^2$. By differentiation with respect to Z, $d(U^2) = (2/w)(L^w - eZ^w)^{2/w-1}(-e\,w\,Z^{w-1})\,dZ - 2(Z + L \cos v_0)dZ$, whence $d(U^2)/2 = -[(L^w - eZ^w)^{2/w-1} eZ^{w-1} + (Z + L \cos v_0)]\,dZ$. When $R = Q$, $dZ = 0$, and a minimum of U^2 and hence of U occurs, as is obviously the fact.

When v_0 is obtuse, a maximum of U^2 and hence of U occurs if $(Z + L \cos v_0) = -(L^w - eZ^w)^{2/w-1} eZ^{w-1}$. Since $(L^w - eZ^w) = R^w$, it is positive; Z and e are also positive. The maximum occurs, then, when $L \cos v_0$, which is negative, numerically exceeds Z.

In the ellipse, when $w = 1$, $L = A + C$, $e = C/A$, and $v_0 = 180°$. The condition for a maximum of U becomes $Z - A - C = -[(A+C) - CZ/A]\,C/A$. Multiplying by A^2 and reducing, $(A^2 - C^2)\,Z = A^2(A + C) - AC(A + C)$, whence $Z = A$, and the maximum of U is B, as is otherwise known.

When $w = 2$, $v_0 = 90°$ and $L \cos v_0 = 0$; the condition for a maximum of U is $Z = -eZ$; hence $Z = 0$, and the maximum of U is A, as is otherwise known.

When $w > 2$, $(Z + L \cos v_0)$ is positive, and no maximum of U, in the strict sense of that word, will occur. The greatest value of U will be $L \sin v_0$.

VIII.

Area included by the curve and its axes.

The value of U, as above, is $\sqrt{[(L^w - eZ^w)^{2/w} - (Z + L \cos v_0)^2]}$. The integration of UdZ between the limits $Z = K$ and $Z = 0$ should give twice the area enclosed by the curve and its axes, with the addition of the rectangle $L^2 \sin v_0 \cos v_0$. When $w = 1$ or $w = 2$, either $\sin v_0$ or $\cos v_0$ has the value 0, and this rectangle disappears. The integration, in other cases, if not directly practicable may be effected by the customary indirect methods. It is not here discussed.

When $w = 1$, $U^2 = [(A + C) - CZ/A]^2 - [Z - (A + C)]^2$, that is $(A+C)^2 - 2(A+C)CZ/A + C^2Z^2/A^2 - Z^2 + 2(A+C)Z - (A+C)^2$, and after reduction $[(C^2 - A^2)Z^2 - 2A(A + C)(C - A)Z]/A^2 = (2AB^2Z - B^2Z^2)/A^2$.

In the circle with radius A, the squared perpendicular which corresponds to U is expressed by $A^2 - (A - Z)^2 = 2AZ - Z^2$; so that U^2 has the ratio B^2/A^2 to this squared perpendicular, and U has the ratio B/A to the perpendicular itself, whence we find the elliptic area in the usual way, when apastron occurs.

When $w = 2$, $U^2 = A^2 - (A^2 - B^2)Z^2/B^2 + Z^2 = [A^2B^2 - A^2Z^2 + B^2Z^2 - B^2Z^2]/B^2$, which becomes $A^2(B^2 - Z^2)/B^2$. In the circle with radius B, the squared perpendicular which corresponds to U is $B^2 - Z^2$, so that U has the ratio A/B to that perpendicular, and the elliptic area is found as before.

IX.

Value of the radius vector at which the forces are equal.

It is now desired to find the value of the radius vector P. When $R = P$, the variation of R with respect to time is at a maximum. Denoting time by t, we now require a maximum value for $dR/2dt$, which may be expressed either as $dR/U\,dZ$ or as dR/R^2dv. In the first case, the independent variable will be Z; in the second, $\cos v$ may be more convenient than v itself.

A maximum of dR^2/U^2dZ^2 will occur with that of dR/UdZ. In the differentiation required to determine this maximum, it must be noticed that U^2 increases while Z diminishes, and hence that $d(U^2)/dZ$ is negative.

We have first to express dR^2/dZ^2 in terms of Z by means of the equation $R^w = L^w - eZ^w$, from which $wR^{w-1}\,dR = -weZ^{w-1}\,dZ$, and $-dR/dZ = eZ^{w-1}/R^{w-1} = eZ^{w-1}/(L^w - eZ^w)^{(w-1)/w}$, whence $dR^2/dZ^2 = e^2Z^{2(w-1)}/(L^w - eZ^w)^{2(w-1)/w}$.
Since $U^2 = (L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2$,
$dR^2/U^2dZ^2 = e^2Z^{2(w-1)}/[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2](L^w - eZ^w)^{2(w-1)/w}$.

We may omit the factor e^2, which will not affect the result.

The denominator of the expression obtained for dR^2/U^2dZ^2 is $U^2R^{2(w-1)}$. It is always positive, and $U = 0$ only at periastron. We have therefore only to consider the numerator of the required differential expression. It will consist of three terms, the differential of dR^2 multiplied by U^2dZ^2, the differential of U^2 multiplied by dR^2 and dZ^2, and the differential of dZ^2 multiplied by dR^2 and U^2. The second and third of these terms will have their original signs reversed.
The first term is $2(w-1)Z^{2(w-1)-1}\,dZ[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]$ $(L^w - eZ^w)^{2(w-1)/w}$. The second is $-Z^{2(w-1)}\,dZ[(L^w - eZ^w)^{2(w-1)/w}]$ $[2/w)\,(L^w - eZ^w)^{2/w-1}\,weZ^{w-1} + 2\,(Z + L\cos v_0)]$. The third is $Z^{2(w-1)}$ $[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]\ [(2\,(w-1)/w)\ (L^w - eZ^w)^{2(w-1)/w-1}$ $weZ^{w-1}dZ]$.

We may remove the factor $2Z^{2(w-1)-1}\,dZ$, which indicates minima at periastron and apastron. The remaining factors may be reduced to the form

$$(w-1)\,[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]\,(L^w - eZ^w)^{2(w-1)/w} \tag{1}$$

$$-Z\,[(L^w - eZ^w)^{2(w-1)/w}]\,[(L^w - eZ^w)^{2/w-1}\,eZ^{w-1} + (Z + L\cos v_0)] \tag{2}$$

$$(w-1)Z\,[(L^w - eZ^w)^{2/w} - (Z + L\cos v_0)^2]\,[(L^w - eZ^w)^{2(w-1)/w-1}\,eZ^{w-1}] \tag{3}$$

the sum of these three terms being 0 when the maximum is reached. For any given value of w, the corresponding value of Z at the maximum may be found, indirectly if not directly. From this value of Z, the value of P is found by means of the equation of the curve.

When $w = 1$, the expression to be differentiated becomes $1/U^2$, and the proposed operation is not directly practicable. But in this case dR/dZ is constant, so that we may regard R as the independent variable. Since $U = R\sin v$, $dU < dR$ except when $U = R$, that is, when R is perpendicular to the axis of periastron, and an extreme value of dU will then occur. The semiparameter of the conic section is otherwise known to be the radius vector P, at which the outward and inward forces are equal.

When $w = 2$, the three terms found above, the sum of which is 0 when the maximum is reached, become

$$(A^2-eY^2-Y^2)(A^2-eY^2) = A^4-(1+e)A^2Y^2-eA^2Y^2+e(1+e)Y^4 \quad (1)$$
$$-(A^2-eY^2)(eY^2+Y^2) = -(1+e)A^2Y^2+e(1+e)Y^4 \quad (2)$$
$$(A^2-eY^2-Y^2)eY^2 = +eA^2Y^2-e(1+e)Y^4 \quad (3)$$

since then $L = A$, the semi-axis major, and $Z = Y$. The value of e in this case was found above to be $(A^2 - B^2)/B^2$, in which B denotes the semi-axis minor.

The equation determining the value of Y when $R = P$ is

$$A^4 - 2(1+e)A^2Y^2 + e(1+e)Y^4 = 0.$$

The corresponding results from the use of R^2dv as an equivalent for $2dt$ will next be deduced from the equation $L^w = R^w + eZ^w = R^w + eR^w \cos^w v$, whence $R^w = L^w/(1 + e\cos^w v)$. To find dR, we have $wR^{w-1}\,dR = -L^w ew \cos^{w-1} v\, d\cos v/(1 + e\cos^w v)^2$, and $R^{w-1} = L^{w-1}/(1 + e\cos^w v)^{(w-1)/w}$, so that

$$dR = -d\cos v[eL\cos^{w-1} v(1+e\cos^w v)^{(w-1)/w-2}].$$

Also $1/R^2 = (1 + e\cos^w v)^{2/w}/L^2$, and $1/dv = -\sin v/d\cos v$.

Hence $dR/R^2dv = e\sin v\, L^{-1}\cos^{w-1}v(1+e\cos^w v)^{(w-1)/w-2}(1+e\cos^w v)^{2/w}$, in which we may omit the factor eL^{-1}, and differentiate the remaining expression with respect to $\cos v$ to obtain the condition for the maximum.

It will here be considered sufficient to examine the results thus obtained for the two cases $w = 1$ and $w = 2$.

When $w = 1$, $dR/R^2dv = e\sin v/L$, which obviously has a maximum when $v = 90°$, confirming the result less directly found above.

When $w = 2$, $dR/R^2dv = e\sin v\cos v/L\sqrt{(1 + e\cos^2 v)}$. Omitting e/L, and squaring, we require a maximum for $(1 - \cos^2 v)\cos^2 v/(1 + e\cos^2 v)$. The squared denominator is always positive, and we need to consider only the numerator of the differentiated expression. It will be the sum of three terms

$$-2\cos^3 v\,(1 + e\cos^2 v) \quad (1)$$
$$2\cos v\,(1 + e\cos^2 v)(1 - \cos^2 v) \quad (2)$$
$$-2e\cos^3 v\,(1 - \cos^2 v) \quad (3)$$

The minimum occurs when $\cos v = 0$, and in finding the maximum the factor $2 \cos v$ may be omitted.

$$
\begin{array}{rcll}
-\cos^2 v(1 + e\cos^2 v) & = & -\cos^2 v \qquad - e\cos^4 v & (1)\\
(1 - \cos^2 v)(1 + e\cos^2 v) & = & 1 - \cos^2 v + e\cos^2 v - e\cos^4 v & (2)\\
-e\cos^2 v(1 - \cos^2 v) & = & \qquad - e\cos^2 v + e\cos^4 v & (3)
\end{array}
$$

and the required equation is $1 - 2\cos^2 v - e\cos^4 v = 0$, which may also be expressed as $\sin^2 v - \cos^2 v - e\cos^4 v = 0$, the value of e being $(A^2 - B^2)/B^2$ as before. It appears from this equation that as e approaches 0, so that the orbit becomes nearly circular, the value of v, when $R = P$, approaches 45°; and that it increases toward 90° as the ellipse elongates.

We may also refer the axial measurements to the major axis of the ellipse by means of the equation $R^2 - B^2 = e^2X^2$, in which e denotes the eccentricity of the ellipse, so that $e^2 = (A^2 - B^2)/A^2$. Since $X = R \sin v$, $R^2 = B^2/(1 - e^2 \sin^2 v)$. The value of R^2dv is now $R^2d \sin v/\cos v = B^2d \sin v/\cos v\,(1 - e^2 \sin^2 v)$. From the value of R^2 just found, $2RdR = 2B^2e^2 \sin v\, d \sin v/(1 - e^2 \sin^2 v)^2 = 2BdR/\sqrt{(1 - e^2 \sin^2 v)}$, and $dR = Be^2 \sin v\, d \sin v/(1 - e^2 \sin^2 v)^3/^2$, whence
$dR/R^2dv = e^2 \sin v \cos v/B\sqrt{(1 - e^2 \sin^2 v)}$. Omitting e^2/B, a maximum is required for $\sin v \cos v/\sqrt{(1 - e^2 \sin^2 v)}$ or for its square $\sin^2 v \cos^2 v/(1 - e^2 \sin^2 v)$. The numerator of the differential coefficient is the sum of three terms.

$$
\begin{array}{rl}
2 \sin v(1 - \sin^2 v)\,(1 - e^2 \sin^2 v) & (1)\\
-\,2 \sin^3 v(1 - e^2 \sin^2 v) & (2)\\
2\,e^2 \sin^3 v(1 - \sin^2 v) & (3)
\end{array}
$$

The minimum occurs when $\sin v = 0$. Omitting the factor $2 \sin v$

$$
\begin{array}{rcll}
(1 - \sin^2 v)\,(1 - e^2 \sin^2 v) & = & 1 - \sin^2 v - e^2 \sin^2 v + e^2 \sin^4 v & (1)\\
-\sin^2 v \quad (1 - e^2 \sin^2 v) & = & -\sin^2 v \qquad + e^2 \sin^4 v & (2)\\
e^2 \sin^2 v \quad (1 - \sin^2 v) & = & \qquad + e^2 \sin^2 v - e^2 \sin^4 v & (3)
\end{array}
$$

and the required equation is $\cos^2 v - \sin^2 v + e^2 \sin^4 v = 0$, in which $e^2 = (A^2 - B^2)/A^2$.

We previously found $\sin^2 v - \cos^2 v - (A^2 - B^2)\cos^4 v/B^2 = 0$. The sum of the two equations is $(A^2 - B^2) \sin^4 v/A^2 - (A^2 - B^2) \cos^4 v/B^2 = 0$, whence $\sin^2 v : \cos^2 v = A : B$, and since $\sin^2 v = X^2/P^2$ and $\cos^2 v = Y^2/P^2$, $X^2 : Y^2 = A : B$ when $R = P$. This equation may

also be derived from the employment of UdZ for the equivalent of $2dt$. It is needless to give the details of this process.

When $R = P$, $X = P \sin v$ and $Y = P \cos v$. It has just appeared that $A^2 = B^2 \tan^4 v$. Hence, from the equation $B^2X^2 + A^2Y^2 = A^2B^2$, after division by B^2, $P^2 (\sin^2 v + \tan^4 v \cos^2 v) = A^2$. Since $\tan^4 v \cos^2 v = \sin^4 v/\cos^2 v$, $\sin^2 v + \tan^4 v \cos^2 v = \sin^2 v(1 + \tan^2 v) = \sin^2 v/\cos^2 v = \tan^2 v$, and $P^2 \tan^2 v = A^2$. Accordingly, while the value of v increases as the ellipse elongates, as appeared above, the ratio A/P also increases.

Since $\tan^2 v = X^2/Y^2 = A/B$, when $R = P$, $AP^2/B = A^2$, and $P^2 = AB$. Also $\sin^2 v + \cos^2 v : \cos^2 v = (A + B) : B$, and $\cos^2 v = B/(A + B)$. Similarly, $\sin^2 v = A/(A + B)$. The position and length of the radius vector P, when $w = 2$, are thus determined.

As stated in section VI, cases in which $w<1$ are not minutely examined in the present discussion. But it may be remarked that since, when $w = 1$, the radius vector P is perpendicular to the axis of periastron, we may expect P to make with that axis an angle v, greater than 90°, when $w < 1$. If no apastron occurs, the curve will not reach this direction of P at all, and the outward force will always be in excess; if R ever assumes the value P, it will subsequently attain the value L, for which value the angle v_0 will exceed 180°.

X.

Relative times of revolution in closed orbits of the same system.

Since the momentary tendency to transverse motion is expressed by the square root of the product of the radius vector and twice the outward force, as appeared above in section I, and since the outward force is expressed by mP^{n+3}/R^3, as in section III, the momentary transverse motion is expressed by $\sqrt{(2mP^{n+3}/R^2)}$. The corresponding increase of area is the product of this quantity by half the radius vector, hence $\sqrt{(mP^{n+3}/2)}$. The ratio of momentary increments of area in two orbits belonging to the same system is therefore the ratio of their respective values of $\sqrt{P^{n+3}}$. Let these values be denoted by $\sqrt{(P_1^{n+3})}$ and $\sqrt{(P_2^{n+3})}$, and let the corresponding areas traversed by the radii vectores from periastron to apastron be denoted by a_1 and a_2. The corresponding times of these angular movements are directly proportional to the areas and inversely proportional to the momentary increments of area, so that their ratio is $a_1\sqrt{(P_2^{n+3})}/a_2(\sqrt{P_1^{n+3}})$.

The possibility of determining the total areas is discussed above in section VIII.

When $w = 1$, $n + 3 = 1$, and the ratio of the areas in a closed orbit is that of half the products of the major and minor semi-axes. The value of P in this case is B^2/A, the proof of which need not here be repeated. The conclusion follows that the times of revolution are proportional to the square roots of the cubes of the major semi-axes.

When $w = 2$, $n + 3 = 4$, and $\sqrt{(P^{n+3})} = P^2$. The area described by the radius vector from periastron to apastron is one quarter of the area of the ellipse, and apastron always occurs. Hence the ratio of the two areas a_1 and a_2 is that of $A_1 B_1$ and $A_2 B_2$, in which A_1 and A_2, B_1 and B_2, respectively denote the major and minor semi-axes. We have found the value of P^2 in this case to be AB. Hence $a_1\sqrt{(P_2^{n+3})}/a_2\sqrt{(P_1^{n+3})}$ becomes $A_1B_1A_2B_2/A_2B_2A_1B_1 = 1$; that is, when $w = 2$, the same time is required for all orbits of that system. This result is obvious in the case of circular orbits, as Newton states in the first book of the Principia.